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Multi-Hamiltonian Structure of kdv Hierarchy by Drienfeld–Sokolov Formalism

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Conserved quantities and the multi-Hamiltonian structure for the integrable coupled kdv system which is associated with the isospectral flow $[(\sum_{i=0}^{N-1} \epsilon_i\lambda^i)\partial^2 + \sum_{i=0}^{N-1} V_i\lambda^i]\psi = \lambda^N\psi$ are deduced by the Drienfeld–Sokolov formalism.

1. INTRODUCTION

The study of nonlinear integrable equations plays an important role in theoretical physics. These integrable equations have been solved by the inverse scattering method (Ablowitz *et al.*, 1974; Kruskal *et al.*, 1970), i.e., they admit the zero-curvature representation

$$V_t - U_x + [V, U] = 0 \tag{1}$$

with a given isospectral problem

$$\psi_x = V\psi \tag{2}$$

where λ is the isospectral parameter. With the condition $\lambda_t = 0$ one can relate the given isospectral parameter to a hierarchy of nonlinear evolution equations

$$V_t - U_x^{(n)} + [V, U^{(n)}] = 0 \tag{3}$$

where $\psi_t = U(\lambda)\psi$.

In the theory of integrable systems we search for a symplectic operator B and a sequence of scalar functions H_n such that equation (3) can be written in the Hamilton form

$$V_t = B\frac{\delta H_n}{\delta V} \tag{4}$$

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where $V = V_1, \ldots, V_n$ in the potential contained in the matrix $V = V(V, \lambda)$ and $\delta/\delta V$ stands for the variational derivative. The Hamiltonian H_n constitutes in fact an infinite number of conserved densities in the hierarchy. Various techniques have been developed to calculate H_n (Gui-Zhang, 1989; Fuchssteiner and Fokas, 1981). A system of evolution equations is said to be bi-Hamiltonian if there exist two symplectic operators B_1 and B_2 and two Hamiltonian H_1 and H_2 such that

$$V_t = B_1 \frac{\delta H_1}{\delta V} = B_2 \frac{\delta H_2}{\delta V} \tag{5}$$

The recursion operator R of the bi-Hamiltonian system is given formally as

$$R = B_2(B_1)^{-1} \tag{6}$$

In general the recursion operator is integrodifferential and the resulting Hamiltonian flows can be mapped directly into each other by use of it.

We obtain the multi-Hamiltonian structure and recursion operator for a coupled kdv equation (Antonowicz and Fordy, 1987) by the Drienfeld and Sokolov (1984) formalism. This formalism is based on affine Lie algebra (Kac, 1990).

2. FORMULATION

The Drienfeld–Sokolov formalism is based on the subalgebra decomposition of a Lie algebra and the proper choice of the Heisenberg algebra of the corresponding affine algebra.

A Lie algebra is usually described by a set of generators, say X_i, following a commutation rule,

$$[X_i, X_j] = c_{ijk} X_k$$

which by virtue of the Jacobi identities implies

$$c_{ijk} + c_{jki} + c_{kji} = 0$$

In general one can classify the generators into three groups, one comprising a certain subalgebra H, and $E_{\beta+}$ $E_{\beta-}$, so that the Lie algebra g can be written as

$$g = H + \sum_{\alpha \in \Delta^+} (E_{\alpha+} + E_{\alpha-})$$

Δ^+ is the set of positive roots. The roots are defined via

$$ad(H)(E_\alpha) = \pm\alpha(H) E_\alpha$$

along with the following commutation rules:

$$[H_\alpha, H_\beta] = 0$$

$$[H_\alpha, E_{\pm\beta}] = \pm K_{\alpha\beta}E \pm \beta$$

$$[E_\alpha, E_{-\alpha}] = H_\alpha$$

$$[E_\alpha, E_\beta] = \begin{cases} 0 & \text{if} \quad \alpha + \beta \in \Delta, \quad \alpha \neq \beta \\ N_{\alpha\beta}E_{\alpha+\beta} & \text{if} \quad \alpha \neq \beta \text{ and } \alpha + \beta \not\subset \Delta \end{cases}$$

It is now very much imperative that on integrable two-dimensional problem is a consequence at a zero-curvature equation, which is always embedded in a Lie algebra. Furthermore, the dependence on the spectral parameter is taken care of by changing the Lie algebra to a Kac–Moody or loop algebra. The simplest way to introduce a loop algebra is to introduce the generators

$$T^n_{\alpha+} = \lambda^n E_{\alpha+}, \qquad T^m_{\beta-} = \lambda^m E_{\beta-}$$

$$H^l_\sigma = \lambda^l H_\sigma$$

so that the general set of commutation rules can be written as

$$[X^n_i, X^m_j] = c_{ijk}(n, m)X^{n+m}_k$$

which is an infinite-dimensional Lie algebra. On the other hand, one can also represent a Lie algebra with the help of generators e_{ij} with a unit element at the intersection of its row and its column and zero elsewhere and following the rules

$$e_{ij}e_{kl} = \delta_{ik}e_{il}$$

$$[e_{ij}, e_{kl}] = \delta_{ik}e_{il} - \delta_{il}e_{kj}$$

and H_α, $E_{\beta+}$, and $E_{\beta-}$ can always be expressed as a linear combination of e_{ij}.

The Lax operator is written as

$$L = \frac{\partial}{\partial x} + V + \Lambda \tag{7}$$

with V an upper triangular matrix and

$$\Lambda = I + \lambda e$$

$$= I + \lambda \sum_{i=1}^{k-1} e_{i+1,i}$$

In accord with the prescription of Drienfeld and Sokolov, it is possible to find a transformation

$$e^{U}Le^{-U} = L + [U, L] \rightarrow \frac{1}{2!}[U, [U, L]] + \cdots = -D + V + H \quad (8)$$

Let

$$e^{U}\lambda^{n}e^{-U} = \phi(\lambda^{n}) \quad (9)$$

Then the nonlinear integrable class of equations is given as

$$\frac{dL}{dt} = [L, \phi(\lambda^{n})^{+}] = -[L, \phi(\lambda^{n})^{-}] \quad (10)$$

where the plus (minus) sign indicates that only positive (negative) powers are to be taken. In our problem, we have introduced the λ dependence in V itself, keeping the matrix structure intact. So we have obtained a new nonlinear integrable system whose properties we analyze in this paper.

3. RESULTS

The system of equations we present is associated with the linear spectral problem

$$L\psi = \left[\left(\sum_{i=0}^{N-1} \epsilon_i\lambda^{i}\right)\partial^{2} + \sum_{i=0}^{N-1} V_i\lambda^{i}\right]\psi = \lambda^{N}\psi \quad (11)$$

When $N = 2$ and $\epsilon_1 = 0$, the L operator of the spectral problem reduces to

$$L = -DI - (V_0 + V_1\lambda)\sigma_- + \Lambda \quad (12)$$

where

$$\Lambda = e_{12} + \lambda^{2}e_{21} = \sigma_+ + \lambda^{2}\sigma_- \quad (13)$$

and V_0 and V_1 are nonlinear fields with values in the set of upper or lower triangular matrixes. We search for U in the form

$$U = \sum (u_{1i}\,\sigma_+ + u_{2i}\sigma_- + u_{3i}\sigma_3)\lambda^{-i} \quad (14)$$

It is well known from the basics of Lie algebra that

$$e^{U}Le^{-U} = L + [U, L] + \frac{1}{2!}[U, [U, L]] + \cdots$$
$$= -DI + \lambda^{2}\sigma_- + \sigma_+ + \sum H_{1i}\sigma_+\lambda^{-i} + \sum H_{2i}\sigma_-\lambda^{2-i} \quad (15)$$

where the H_i are nothing but Hamiltonians of the system. From these equations we have

$$U = (U_{1,-1}\sigma_+ + u_{2,-1}\sigma_- + u_{3,-1}\sigma_3)\lambda^{-1} + (u_{1,-2}\sigma_+ + u_{2,-2}\sigma_- + u_{3,-2}\sigma_3)\lambda^{-2}$$
$$+ (u_{1,-3}\sigma_+ + u_{2,-3}\sigma_- + u_{3,-3}\sigma_3)\lambda^{-3} + \cdots$$
$$= \left(\frac{v_1x}{8}\sigma_- + \frac{v_1}{4}\sigma_3\right)\lambda^{-1} + \left[\left(-\frac{1}{8}v_1v_{1x} + \frac{1}{8}v_{0x}\right)\sigma_- + \left(-\frac{1}{8}v_1^2 + \frac{v_0}{4}\right)\sigma_3\right]\lambda^{-2}$$
$$+ \left[\frac{v_{1x}}{8}\sigma_+ + \left(-\frac{1}{32}v_{1xxx} - \frac{1}{8}v_{0x}v_1 - \frac{1}{8}v_0v_{1x} + \frac{9}{64}v_1^2v_{1x}\right)\sigma_-\right.$$
$$\left. + \left(-\frac{1}{16}v_{1xx} - \frac{1}{4}v_0v_1 + \frac{1}{2}v_1^3\right)\sigma_3\right]\lambda^{-3} + \cdots \tag{16}$$

This gives the following expression for $\phi(\lambda^2)^{3+}$ according to expression (7):

$$\phi(\lambda^2)^{3+} = \{\lambda^2 - 2u_{3,-1}\lambda + (-2u_{3,-2} + 2u_{3,-1}^2) + \cdots\}\sigma_+$$
$$+ \left\{-\lambda^4 + \left(-2u_{3,-4} - 4u_{3,-1}u_{3,-2} - \frac{4}{3}u_{3,4}^3\right)\lambda^3 + \cdots\right\}\sigma_-$$
$$+ \{(u_{1,-2} + u_{2,-1})\lambda$$
$$+ (u_{1,-4} + u_{2,-4} + u_{1,-3}u_{3,4} - u_{2,-1}u_{3,-1}) + \cdots\}\sigma_3 \tag{17}$$

Next we obtain

$$\frac{dL}{dt} = [\phi(\lambda^2)^{3+}, L] \tag{18}$$

$$\Rightarrow \begin{cases} v_{0t_3} = \left(\frac{1}{4}\partial^3 + v_0\partial + \frac{1}{2}v_{0x}\right)\left(v_0 + \frac{3}{4}v_1^2\right) \\ v_{1t_3} = \left(\frac{1}{4}\partial^3 + v_0\partial + \frac{1}{2}v_{0x}\right)v_1 + \left(v_1\partial + \frac{1}{2}v_{1x}\right)\left(v_0 + \frac{3}{4}v_1^2\right) \end{cases} \tag{19}$$

where v_0 is the kdv variable. If v_1 is set equal to zero, then equation (18) reduces to the well-known kdv equation. The first five Hamiltonians are

$$H_1 = \frac{1}{2}v_1$$

$$H_2 = \frac{1}{2}\left(v_0 + \frac{1}{4}v_1^2\right)$$

$$H_3 = \frac{1}{2}\left(\frac{1}{2} v_0 v_1 + \frac{1}{16} v_1^3\right)$$

$$H_4 = \frac{1}{2}\left(\frac{1}{4} v_0^2 + \frac{3}{8} v_0 v_1^2 + \frac{5}{64} v_1^4 + \frac{1}{16} v_{1x}^2\right)$$

$$H_5 = \frac{1}{2}\left(-\frac{1}{4} v_1^2 v_{1xx} - \frac{1}{16} v_1 v_{1x}^2 + \frac{1}{16} v_1 v_{0xx} + \frac{1}{16} v_0 v_{1xx} + \frac{5}{16} v_0 v_1^3 + \frac{3}{8} v_0^2 v_1 + \frac{1}{128} v_1^5\right)$$

Equation (18) can be written in the Hamiltonian form

$$\begin{bmatrix} v_{0t_3} \\ v_{1t_3} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \frac{1}{4}\partial^3 + v_0\partial + \frac{1}{2} v_{0x} & 0 \\ 0 & \partial \end{bmatrix}\begin{bmatrix} v_0 + \frac{3}{4} v_1^2 \\ \frac{3}{2} v_0 v_1 + \frac{5}{8} v_1^3 + \frac{1}{4} v_{1xx} \end{bmatrix} \tag{20}$$

$$\Rightarrow v_{t_3} = B_3\, \delta H_3 = B_2\, \delta H_4 = B_1\, \delta H_5 \tag{21}$$

where

$$B_1 = \frac{1}{4}\begin{bmatrix} -\frac{1}{2}(v_1\partial + \partial v_1) & \partial \\ \partial & 0 \end{bmatrix} \tag{22a}$$

$$B_2 = \frac{1}{4}\begin{bmatrix} \frac{1}{4}\partial^3 + \frac{1}{2}(v_0\partial + \partial v_0) & 0 \\ 0 & \partial \end{bmatrix} \tag{22b}$$

$$B_3 = \frac{1}{4}\begin{bmatrix} 0 & \frac{1}{4}\partial^3 + \frac{1}{2}(v_0\partial + \partial v_0) \\ \frac{1}{4}\partial^3 + \frac{1}{2}(v_0\partial + \partial v_0) & \frac{1}{2}(v_1\partial + \partial v_1) \end{bmatrix} \tag{22c}$$

So we have the obtained multi-Hamiltonian structure of the coupled kdv system (5).

When $v_1 \to 0$, the coupled system reduces to

$$v_{0t_3} = B_2\, \delta H_4 = B_1\, \delta H_5 \tag{23}$$

where B_2 and B_1 are the second and first symplectic operators of the kdv system, respectively.

A bi-Hamiltonian system which enjoys a compatible pair of Hamiltonian structure is known (in general) to an infinite hierarchy of bi-Hamiltonian system. The resulting Hamiltonian flows can be mapped directly into each other by use of the recursion operator R, which is given formally as the "quotient" of the two Hamiltonian structures:

$$R = B_2(B_1)^{-1}$$

$$= \begin{bmatrix} 0 & \left\{\frac{1}{4}\partial^3 + \frac{1}{2}(v_0\partial + \partial v_0)\right\}\partial^{-1} \\ 1 & \left\{-\frac{1}{2}(v_1\partial + \partial v_1)\right\}\partial^{-1} \end{bmatrix} \tag{24}$$

4. CONCLUSION

In the above analysis we have obtained the multi-Hamiltonian structure of the coupled kdv system by using the Drienfeld–Sokolov formalism. We have also obtained the recursion operator of the system.

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